Algebraic Models for Transdisciplinarity

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Abstract

We discuss about transdisciplinarity, interdisciplinarity and pluridisciplinarity, and unification theories. Some duality theorems related to these topics are presented. We give plenty of algebraic details.

1 Introduction

Transdisciplinarity ([1, 2, 3]) is a relatively young approach. It concerns itself with what is between disciplines, across different disciplines, and beyond all disciplines. Its goal is the understanding of the present world.

Written at a transdisciplinary level, the current paper abides somewhere between Epistemology and Abstract Algebra.

Attempting to unify two algebraic structures which looked different (although related by duality) the author obtained new objects in [4]. They are solutions for a celebrated equation traversing Statistical Mechanics, Theoretical Physics ([5]), Knot Theory ([6]), Quantum Groups ([7]), etc.

In this paper we argue that the above unification has many similarities with the following picture.

T r a n s d i s c i p l i n a r i t y

$\uparrow \qquad\qquad\qquad\qquad \uparrow$

Interdisciplinarity **Pluridisciplinarity**

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The remaining of this paper is organized as follows. In the next section we discuss about an algebraic model for transdisciplinarity (viewed as a "large approach" which includes both interdisciplinarity and pluridisciplinarity). In Section 3 we present algebraic details: the category of Yang-Baxter structures, examples of such structures, full and faithful embbedings from the categories of (co)algebra structures to the category of Yang-Baxter structures, etc. Section 4 is an account about duality extensions.

The purposes of this paper are the following.

1. Clarifying the concepts of transdisciplinarity, interdisciplinarity and pluridisciplinarity for the interested mathematicians.

2. A deeper understanding of transdisciplinarity at an epistemologic level. For example, disciplinarity can be viewed as a particular case of both interdisciplinarity and pluridisciplinarity (lying in their intersection).

3. The presentation of (a new) duality extension(s).

2 An Algebraic Model

Let us start with the diagram showing that the transdisciplinarity extends both the interdisciplinarity and the pluridisciplinarity.

$$\begin{array}{ccc} \multicolumn{3}{c}{\textbf{T r a n s d i s c i p l i n a r i t y}} \\ \uparrow & & \uparrow \\ \textbf{Interdisciplinarity} & <---> & \textbf{Pluridisciplinarity} \end{array}$$

Let us observe some properties of the above concepts.

The *interdisciplinarity* generates new disciplines. For example, the transfer of mathematical models in physics has generated the mathematical-physics. Using the mathematical formalism, for two disciplines D_1 and D_2, the interdisciplinarity associates D (a discipline which in not necessarily D_1 or D_2). Thus, interdisciplinarity resembles the algebraic notion of operation. (For example, the matrix multiplication of two square matrices is a product or an operation.) Also, we could think about an "empty discipline", Θ, such that for Θ and D the interdisciplinarity associates D (the "empty discipline" behaves like the unity in an algebra).

The *pluridisciplinarity* refers to the study of one object from one discipline by the use of other disciplines. Thus, taking an object from one discipline, $o \in D$, we first consider its images in other disciplines, $o_i \in D_i$, and then return to D (with some conclusions). We now can see similarities with the notion of coalgebra. We can say that pluridisciplinarity has a co-operation and a counit. (Examples of co-operations are: maps which associate to a set some partitions of it, functions which for some numbers associate their divisors, etc.)

Referring to Abstract Algebra, or more precisely to Hopf Algebra Theory, we consider now structures with multiplications and comultiplications.

An *algebra structure* has a multiplication which assigns one element to any of its two elements. It also has a unity.

A *coalgebra structure*, by its comultiplication, has the property of assigning many elements to an element. It also has a counity. (Notice an equation related to the above commentary on pluridisciplinarity: $(\varepsilon \otimes Id) \circ \Delta = Id$.)

Attempts to unify the category of algebras and the category of coalgebras lead to structures which have as inputs two elements and as outputs (a set of) two elements, called Yang-Baxter structures (see [8]).

$$\begin{tikzcd}
\textbf{Yang} - \textbf{Baxter} & & \textbf{structures} \\
\mathbf{k - alg} \arrow[u] & < - - - > & \mathbf{k - coalg} \arrow[u]
\end{tikzcd}$$

3 Algebraic Insight

Throughout the remaining of this paper k is a field such that char $k \neq 2$.

We use the following terminology concerning the Yang-Baxter equation (see [9]).

For V a k-space, we denote by $I : V \to V$ and $I_2 : V \otimes V \to V \otimes V$ the identity maps.
Let T be the twist map $T : V \otimes V \to V \otimes V$, $\quad T(v \otimes w) = w \otimes v$.
Let $R : V \otimes V \to V \otimes V$ be a k-linear map. We use the following notations:
$R^{12} = R \otimes I, R^{23} = I \otimes R, R^{13} = (I \otimes T)(R \otimes I)(I \otimes T)$.

Definition 3.1 *An invertible k-linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator (or simply a YB operator) if it satisfies the equation*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1}$$

The following terminology and theorems concerning Hopf algebras (see [10]) are needed.

For any vector spaces V and W, we denote by $i_{V,W} : V^* \otimes W^* \to (V \otimes W)^*$ the natural injection defined by $i_{V,W}(v^* \otimes w^*)(v \otimes w) = v^*(v)w^*(w)$ for any $v^* \in V^*, w^* \in W^*, v \in V, w \in W$. If V and W are finite dimensional spaces, then $i_{V,W}$ is a bijection.

If (C, Δ, ε) is a k-coalgebra, then the dual $C^* = Hom_k(C, k)$ has an algebra structure as follows: $M : C^* \otimes C^* \to (C \otimes C)^* \to C^*, \quad M = \Delta^* \circ i_{C,C}$
$u : k \to k^* \to C^*$, $\quad u = \varepsilon^* \circ \eta,$ $\quad$ where $\eta(\alpha 1_k) = \alpha 1_{k^*}$.

If (A, M, u) is a finite dimensional k-algebra, then the dual $A^* = Hom_k(A, k)$ has a coalgebra structure as follows: $\Delta : A^* \to (A \otimes A)^* \to A^* \otimes A^*$, $\quad \Delta = i^{-1}_{A,A} \circ M^*$
$\varepsilon : A^* \to k^* \to k$, $\quad \varepsilon = \eta^{-1} \circ u^*$.

Definition 3.2 *We define the category* **YB str** *(respective* **f.d.YB str***) whose objects are 4-tuples* $(V, \varphi, e, \varepsilon)$ *, where*
i) $\quad V$ *is a (finite dimensional) k-space;*
ii) $\quad \varphi : V \otimes V \to V \otimes V$ *is a YB operator;*
iii) $\quad e \in V$ *such that* $\varphi(x \otimes e) = e \otimes x, \ \varphi(e \otimes x) = x \otimes e \ \ \forall x \in V$ *;*
iv) $\varepsilon \in V \to k$ *is a* $k-map$ *such that* $(I \otimes \varepsilon) \circ \varphi = \varepsilon \otimes I, \quad (\varepsilon \otimes I) \circ \varphi = I \otimes \varepsilon$.

A morphism $f : (V, \varphi, e, \varepsilon) \to (V', \varphi', e', \varepsilon')$ *in the category* **YB str** *is a k-linear map* $f : V \to V'$ *such that:*
v) $\quad (f \otimes f) \circ \varphi = \varphi' \circ (f \otimes f)$;
vi) $\quad f(e) = e'$;
vii) $\quad \varepsilon' \circ f = \varepsilon$.

Remark 3.3 *The following are examples of objects from the category* **YB str***:*

(i) Let $R : V \otimes V \to V \otimes V$ *is a YB operator. Then* $(V, R, 0, 0)$ *is an object in the category* **YB str***.*

(ii) Let V *be a two dimensional k-space generated by the vectors* e_1 *and* e_2 *. Then* (V, T, e_1, e_2^*) *is an object in the category* **f.d. YB str***.*

Theorem 3.4 *([11]) i) There exists a functor:*

$$F : \mathbf{k-alg} \longrightarrow \mathbf{YB\ str}$$

$(A, M, u) \mapsto (A, \varphi_A, u(1) = 1_A, 0 \in A^*)$ $\quad$ *where* $\varphi_A(a \otimes b) = ab \otimes 1 + 1 \otimes ab - a \otimes b$.

Any k-algebra map f is simply mapped into a k-map.

ii) F is a full and faithful embbeding.

Theorem 3.5 *([11]) i) There exists a functor:*

$G : \mathbf{k-coalg} \longrightarrow \mathbf{YB\ str}$

$(C, \Delta, \varepsilon) \mapsto (C, \psi_C, 0 \in C, \varepsilon \in C^*)$ *where* $\psi_C = \Delta \otimes \varepsilon + \varepsilon \otimes \Delta - I_2$.
Any k-coalgebra map f is simply mapped into a k-map.

ii) G is a full and faithful embedding.

The above results can be stated in the following diagram

$$\begin{tikzcd}
& \mathbf{Yang-Baxter\ \ structures} & \\
\mathbf{k-alg} \arrow[ur, "F"] & < - - - > & \mathbf{k-coalg} \arrow[ul, "G"']
\end{tikzcd}$$

4 Duality Extensions and Applications

We attempt to explain the mathematical importance (and meaning) of the unifications presented before. Thus, we present some duality extensions.

The Pontryagin duality theorem refers to the duality between the category of compact Hausdorff Abelian groups and the opposite category of discrete Abelian groups. The Pontryagin-van Kampen duality theorem extends this duality to all locally compact Hausdorff Abelian topological groups (see [12]). This can be seen bellow, in the following diagram:

$$\begin{tikzcd}
\mathbf{LCA} \arrow[r, shift left, "(-,T)"] & \mathbf{LCA^{opp}} \arrow[l, shift left, "(-,T)"] \\
\mathbf{Cpct.} \arrow[u] \arrow[r, shift left, "(-,T)"] & \mathbf{Disc.^{opp}} \arrow[u] \arrow[l, shift left, "(-,T)"]
\end{tikzcd}$$

Taking the Pontryagin-van Kampen duality theorem as a model, we posed the following question: "Is it be possible to extend, the duality between finite dimensional algebras and coalgebras (in the same spirit) ?" We gave a positive answer by constructing an extension for the duality of finite dimensional algebras and coalgebras.

We extended the duality between finite dimensional algebras and coalgebras to the category **f.d. YB str**. This can be seen below, in the following diagram:

$$\begin{tikzcd}
\mathbf{f.d.\ YB\ str} \arrow[r, shift left, "D=(\ )^*"] & \mathbf{f.d.\ YB\ str^{opp}} \arrow[l, shift left, "D=(\ )^*"] \\
\mathbf{f.d.\ k-alg} \arrow[u, "F"] \arrow[r, shift left, "(\ )^*"] & \mathbf{f.d.\ k-coalg^{opp}} \arrow[u, "G"'] \arrow[l, shift left, "(\ )^*"]
\end{tikzcd}$$

Formally, the above diagram is contained in the following theorem.

Theorem 4.1 *([11]) (***Duality Theorem***)*

i) The following is a duality functor: $\mathbf{D} : \mathbf{f.d.\ YB\ str} \longrightarrow \mathbf{f.d.\ YB\ str^{op}}$
$(V,\, \varphi,\, e,\, \varepsilon) \mapsto (V^*,\, i^{-1}_{V,V} \circ \varphi^* \circ i_{V,V},\, \varepsilon,\, \zeta_e)$ *where* $\zeta_e : V^* \to k,\ \zeta_e(g) = g(e) \quad \forall g \in V^*$.
Note that: $D(f) = f^*,$ *for* $f : (V,\, \varphi,\, e,\, \varepsilon) \to (V',\, \varphi',\, e',\, \varepsilon')$.

ii) The following relations hold:
$D(\,(A,\, \varphi_A,\, 1_A,\, 0)\,) \;=\; (A^*,\, \psi_{A^*},\, 0,\, \zeta_{1_A})$
$D(\,(C,\, \psi_C,\, 0,\, \varepsilon)\,) = (C^*,\, \varphi_{C^*},\, \varepsilon = 1_{C^*},\, 0)$

Remark 4.2 *The constructions related to the extension of the duality between finite dimensional algebras and coalgebras have many applications. Some operators like* φ_A *appeared in noncommutative descent theory (see [13]). Independently, operators* φ_A *and* ψ_C *were constructed in order to capture the common piece of information encapsulated in the algebra and coalgebra structures (see [4]). Later, [14] produced larger classes of Yang-Baxter operators associated to (co)algebra structures. These can be used to construct FRT bialgebras (see [7]) and knot invariants (see [6]). A generalization of these constructions producing Yang-Baxter operators of Hecke type from entwining structures is presented in [15]; it is also shown a connection between entwining structures and Yang-Baxter systems. Using the ideas from [14] and [4], [16] produced solutions to the two parameter form of the quantum Yang-Baxter equation.*

Following [15], there is new way to extend the the duality between finite dimensional (co)algebras. This is presented without proof below.

$$\begin{tikzcd}
\textbf{f.d. Entwining str} \arrow[r, shift left, "D=(\ )^*"] & \textbf{f.d. Entwining str}^{\textbf{opp}} \arrow[l, shift left, "D=(\ )^*"] \\
\textbf{f.d. k}-\textbf{alg} \arrow[u, "J"] \arrow[r, shift left, "(\ )^*"] & \textbf{f.d. k}-\textbf{coalg}^{\textbf{opp}} \arrow[u, "N"'] \arrow[l, shift left, "(\ )^*"]
\end{tikzcd}$$

Theorem 4.3 *i) There exists a functor:*

$J : \mathbf{k-alg} \longrightarrow$ **Entwining str**

$A \mapsto (k, A, T)$, *where* T *is the twist map.*

Any k-algebra map f *is simply mapped into* $I_k \otimes f$.

ii) J *is a full and faithful embbeding.*

iii) There exists a functor: $N : \mathbf{k-coalg} \longrightarrow$ **YB str**, $C \mapsto (C, k, T)$.

Any k-coalgebra map g *is simply mapped into* $g = g \otimes I_k$.

iv) N *is a full and faithful embbeding.*

v) The restriction to finit dimensional (co)algebras, produces the above diagram, where $D(C,\ A,\ \psi) = (A^*,\ C^*,\ \psi^*)$. *Notice that:* $D(J(A)) = N(A^*)$ *and* $D(N(C)) = J(C^*)$.

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